Two-loop calculation of the turbulent Prandtl number

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The turbulent Prandtl number has been calculated in the two-loop approximation of the ε expansion of the stochastic theory of turbulence. The strikingly small value obtained for the two-loop correction explains the good agreement of the earlier one-loop result with the experiment. This situation is drastically different from other available nontrivial two-loop results, which exhibit corrections of the magnitude of the one-loop term. The reason is traced to the mutual cancellation of additional divergences appearing in two dimensions, which have had a major effect on the results of previous calculations of other quantities.

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I. INTRODUCTION

The method of renormalization group (RG) in the theory of developed turbulence is currently the most developed technical means allowing for reorganization of the straightforward perturbation theory, whose huge expansion parameter at large Reynolds numbers renders it practically useless. At the same time the physical value of the artificial expansion parameter ε introduced in the RG approach is not small either. For some important physical quantities, such as the critical dimension of the velocity and effective viscosity, it is possible to prove with the use of of Galilei invariance of the theory that the corresponding series in ε terminate at the linear terms. Therefore, for such quantities the RG approach yields exact answers coinciding with the prediction of the phenomenological theory of Kolmogorov. For other interesting quantities, such as the Kolmogorov constant, skewness factor, turbulent Prandtl number and the like, the series in ε, however, do not terminate. In this context, it has been often suggested that with the aid of the ε expansions, it is not possible to obtain a sufficiently good estimate of numerical values of these quantities, although, until recently, there were no calculations extending beyond the first order of the perturbation theory (one-loop approximation). The two-loop calculation of the Kolmogorov constant and the skewness factor in the inertial range carried out in Ref. [1] confirmed this pessimistic point of view on the whole: the two-loop contribution turned out to be practically equal to the one-loop contribution, although the trend of change of the quantities calculated was correct i.e., towards the experimental value from the one-loop result.

In Ref. [1] calculations were carried out for space dimensions d different from $d=3$ as well. It turned out that the relative magnitude of the two-loop contribution decreases with the growth of d, and in the limit $d\to\infty$ is of the order of 10% only. At the same time, in the limit $d\to 2$ this contribution grows without limit. Such a behavior of the coefficients of the ε expansion may be related to that their singularities as functions d lie in the region $d\leqslant 2$. The nearest singularity at $d=2$ is connected with the divergence of some graphs in the limit $d\to 2$, which leads to the appearance of poles in $d-2$ in the coefficients of the ε expansion, and it is just these graphs that turn out to be responsible for the large value of the two-loop contribution at $d=3$. This feature gave rise to the hope that summation of these singularities may lead to quantitative improvement of the results of the ε expansion in the real dimension $d=3$. Such a summation was carried out in the framework of the RG method with the aid of the account of the additional UV renormalization of the theory in the vicinity of $d=2$ [2]. In the resulting "improved ε expansion," the low-order terms are calculated in the usual way at $d=3$, while the high-order terms are approximately summed with the account of their leading singularities in $d-2$ (one-loop approximation), then next-to-leading singularities (two-loop approximation), etc. Calculation of the Kolmogorov constant and skewness factor according to this program has demonstrated an essential decrease of the relative impact of the two-loop contribution and led to a fairly good agreement with the experiment [2].

In the present paper we shall analyze to what extent the singularities of the ε expansion show for another important characteristic quantity of turbulent systems, the turbulent Prandtl number. It was calculated in the framework of the RG and the ε expansion in Refs. [3,4] (strictly speaking, in the earliest Ref. [3] the *magnetic* Prandtl number was evaluated) with rather good agreement with experiment [5–7]. We have carried out a two-loop calculation of the Prandtl number in order to check whether this agreement is partially coincidental.

Let us recall that the Prandtl number is the dimensionless ratio of the coefficient of kinematic viscosity ν_0 to the coefficient of thermal diffusivity κ_0. (In the formally identical problem of turbulent diffusion, the ratio of the coefficients of kinematic viscosity and diffusion is called Schmidt number). For systems with strongly developed turbulence the process of homogenization of the temperature is strongly accelerated, which is reflected in the value of the effective or turbulent coefficient of thermal diffusivity. The ratio of the coefficient of turbulent viscosity and the coefficient of turbulent thermal diffusivity is the turbulent Prandtl number. Contrary to its molecular analog, the turbulent Prandtl number is universal, i.e., does not depend on individual properties of the fluid. For the accurate determination of the turbulent Prandtl number a set of conditions is required, especially when calculations are carried out in the two-loop approximation. Therefore, apart from the formulation of the stochastic problem, we shall pay proper attention to this problem as well.

The present paper is organized as follows. In Sec. II we review the main features of the description of passive advection of a scalar quantity in the stochastic theory of turbulence with special emphasis on the careful definition of the turbulent Prandtl number within the model considered. Section III is devoted to the analysis of renormalization and renormalization-group equations of the model. In Sec. IV details of the two-loop calculation are presented. Section V contains analysis of the results and concluding remarks.

II. DESCRIPTION OF THE MODEL

Turbulent mixing of a passive scalar quantity is described by the equation

$$\partial_t \psi + (\varphi_j \partial_j)\psi = \kappa_0 \Delta \psi + f. \tag{1}$$

The field $\psi(\mathbf{x},t)$ in Eq. (1) may have the meaning of both the nonuniform temperature (κ_0 being the coefficient of thermal diffusivity) and concentration of the particles of the admixture (in this case, κ_0 is replaced by the coefficient of diffusion). The field $f(\mathbf{x},t)$ is the source of the passive scalar field. In the stochastic model of turbulence the field of turbulent eddies of the velocity of the incompressible fluid $\varphi_i(\mathbf{x},t)$ satisfies the Navier-Stokes equation with a random force:

$$\partial_t \varphi_i + (\varphi_j \partial_j)\varphi_i = \nu_0 \Delta \varphi_i - \partial_i P + F_i, \tag{2}$$

where $P(t,\mathbf{x})$ and $F_i(t,\mathbf{x})$ are, respectively, the pressure and the transverse external random force per unit mass. For F, a Gaussian distribution with zero mean and the correlation function

$$\langle F_i(t,\mathbf{x})F_j(t',\mathbf{x}')\rangle = \delta(t-t')(2\pi)^{-d}\int d\mathbf{k}P_{ij}(\mathbf{k})d_F(k) \times \exp[i\mathbf{k}(\mathbf{x}-\mathbf{x}')] \tag{3}$$

is assumed. Here, $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projection operator, $d_F(k)$ a function of $k \equiv |\mathbf{k}|$ and parameters of the model, and d the dimension of the coordinate space $\mathbf{x}$.

The stochastic problem (1)–(3) is equivalent to the quantum-field model with the doubled number of fields $\phi \equiv \{\varphi, \psi, \varphi', \psi'\}$ and the action

$$S(\Phi) = \varphi' D_F \varphi'/2 + \varphi'[-\partial_t \varphi + \nu_0 \Delta\varphi - (\varphi\partial)\varphi] + \psi'[-\partial_t\psi + \kappa_0\Delta\psi - (\varphi\partial)\psi + f], \tag{4}$$

in which D_F is the correlation function of the random force (3) and the necessary integrations over $\{t,\mathbf{x}\}$ and summations over vector indices are implied. In model (1)–(4) only correlation functions of the admixture field of the form

$$\langle \psi(\mathbf{x}_1,t_1),\psi(\mathbf{x}_2,t_2)\cdots\psi(\mathbf{x}_n,t_n)\psi'(\mathbf{x}'_1,t'_1),\psi'(\mathbf{x}'_2,t'_2)\cdots\psi'(\mathbf{x}'_n,t'_n)\rangle,$$

with the meaning of multiple response functions are nonvanishing. The simplest of them is determined by the following variational derivative with respect to the source f in Eq. (1):

$$G(\mathbf{x}-\mathbf{x}',t-t') \equiv \langle \psi(\mathbf{x},t)\psi'(\mathbf{x}',t')\rangle|_{f=0} = \left.\frac{\delta\langle\psi(\mathbf{x},t)\rangle}{\delta f(\mathbf{x}',t')}\right|_{f=0}. \tag{5}$$

The nonrandom source field f of the passive scalar has been introduced in action (4) solely to recall relation (5) and its generalizations, and will therefore further be omitted.

Model (4) gives rise to the standard diagrammatic technique with the following nonvanishing bare propagators ($t \equiv t_1 - t_2$):

$$\langle\varphi(t_1)\varphi(t_2)\rangle_0 = \frac{d_F(k)}{2\nu_0 k^2}\exp(-\nu_0 k^2|t|), \tag{6}$$

$$\langle\varphi(t_1)\varphi'(t_2)\rangle_0 = \theta(t)\exp(-\nu_0 k^2 t), \tag{7}$$

$$\langle\psi(t_1)\psi'(t_2)\rangle_0 = \theta(t)\exp(-\kappa_0 k^2 t), \tag{8}$$

in the $(t,\mathbf{k})$ representation. The common factor $P_{ij}(\mathbf{k})$ has been omitted in expressions (6) and (7) for simplicity. Interaction in action (4) corresponds to the three-point vertices $-\varphi'(\varphi\partial)\varphi = \varphi'_i V_{ijs}\varphi_j\varphi_s/2$ with the vertex factor $V_{ijs} = \mathrm{i}(k_j\delta_{is} + k_s\delta_{ij})$, and $-\psi'(\varphi\partial)\psi = \mathrm{i}k_j\psi'\varphi_j\psi$, where $\mathbf{k}$ is the wave vector of fields φ' and ψ'.

Turbulent processes lead to significantly faster attenuation in time of the response functions $\langle\varphi\varphi'\rangle$ and $\langle\psi\psi'\rangle$ than in relations (7) and (8) due to the effective replacement of the molecular coefficients of viscosity and thermal diffusivity by their turbulent analogs. At the same time, however, the simple exponential time dependence is changed as well (and in a different manner for $\langle\varphi\varphi'\rangle$ and $\langle\psi\psi'\rangle$). Therefore, it is necessary to choose a definite way of fixing the ratio of the turbulent transport coefficients; i.e., the Prandtl number (or Schmidt number). Henceforth, we shall use the following definition. Consider the Dyson equations for the response functions in the wave-vector-frequency representation:

$$G^{-1}_{\varphi\varphi'}(k,\omega) \equiv \Gamma_{\varphi\varphi'}(k,\omega) = -\mathrm{i}\omega + \nu_0 k^2 - \Sigma_{\varphi'\varphi}(k,\omega), \tag{9}$$

$$G^{-1}_{\psi\psi'}(k,\omega) \equiv \Gamma_{\psi\psi'}(k,\omega) = -\mathrm{i}\omega + \kappa_0 k^2 - \Sigma_{\psi'\psi}(k,\omega), \tag{10}$$

where Σ are the corresponding self-energy operators, and introduce the inverse effective Prandtl number u_{eff} by the relation

$$u_{eff} \equiv \frac{\Gamma_{\psi\psi'}(k,\omega=0)}{\Gamma_{\varphi\varphi'}(k,\omega=0)}. \tag{11}$$

Further, we shall be interested in the inertial range $L^{-1} \ll k \ll \Lambda$ (here, L is the external scale of turbulence and Λ^{-1} the characteristic length of the dissipating eddies) in which the quantity u_{eff} is independent of the wave number k. The bare value (without turbulence) $u_{eff} = u_0 \equiv \kappa_0/\nu_0$.

III. RENORMALIZATION OF THE MODEL AND THE RG REPRESENTATION

The self-energy operators $\Sigma_{\varphi',\varphi}(k,\omega)$ and $\Sigma_{\psi'\psi}(k,\omega)$ appearing in Eqs. (9) and (10) may be found in model (4) in perturbation theory. However, the expansion parameter turns out to be very large for developed turbulence (for $\Lambda L \gg 1$). The renormalization-group method allows one to carry out a resummation in the straightforward perturbation theory. To apply it, it is necessary to use in relation (3) "the pumping function" $d_F(k)$ of a special form

$$d_F(k) = D_0 k^{4-d-2\varepsilon}. \quad (12)$$

In the infrared region, the power function (12) is assumed to be cut off at wave numbers $k \leqslant m \equiv L^{-1}$. The quantity $\varepsilon > 0$ in Eq. (12) is the formal small expansion parameter in the RG approach with the value $\varepsilon = 2$ corresponding to the physical model.

The usual perturbation theory is a series in powers of the charge $g_0 \equiv D_0/\nu_0^3$ dimensionless at $\varepsilon = 0$ (logarithmic theory). At $\varepsilon \to 0$, ultraviolet divergences are brought about in the graphs of the perturbation theory which show in the form of poles in ε. Due to Galilei invariance of the model, divergences at $d > 2$ are present only in the one-irreducible functions $\langle \varphi\varphi' \rangle$ and $\langle \psi\psi' \rangle$ and are of the form $\varphi'\Delta\varphi$ and $\psi'\Delta\psi$. At $d=2$ the one-irreducible function $\langle \varphi'\varphi' \rangle$ also diverges. For $d > 2$, the renormalized action may be written as

$$S_R(\Phi) = \frac{1}{2}\varphi' D_F \varphi' + \varphi'[-\partial_t\varphi + \nu Z_\nu \Delta\varphi - (\varphi\partial)\varphi] + \psi'[-\partial_t\psi + u\nu Z_\kappa \Delta\psi - (\varphi\partial)\psi].$$

We obtain from action (4), by the multiplicative renormalization of the parameters of the model,

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\varepsilon} Z_g, \quad u_0 = u Z_u, \quad Z_u = Z_\kappa Z_\nu^{-1}, \quad Z_g = Z_\nu^{-3} \quad (13)$$

with two independent renormalization constants Z_ν and Z_κ. The quantities ν, u, and g in Eq. (13) are the renormalized analogs of the coefficient of viscosity, the inverse Prandtl (Schmidt) number and the coupling constant (the charge g being dimensionless). The renormalization mass μ is an arbitrary parameter of the renormalized theory, and the pumping function $d_F(k)$ [Eq. (12)] determining the correlation function of the random force D_F [Eq. (3)] is assumed to be expressed in terms of the renormalized parameters

$$d_F(k) = g_0 \nu_0^3 k^{4-d-2\varepsilon} = g\mu^{2\varepsilon}\nu^3 k^{4-d-2\varepsilon}.$$

The dissipative wave number Λ is determined by g_0 according to the relation $\Lambda = g_0^{1/2\varepsilon}$. It may be also estimated by the quantity μ. Thus, the inertial range we are interested in corresponds to the condition $s \equiv k/\mu \ll 1$.

In the scheme of minimal subtractions (MS) used in the following, the renormalization constants have the form of the Laurent expansion 1+poles in ε

$$Z = 1 + \sum_{k=1}^{\infty} a_k(g,u)\varepsilon^{-k} = 1 + \sum_{n=1}^{\infty} g^n \sum_{k=1}^{n} a_{nk}(u)\varepsilon^{-k}. \quad (14)$$

For Z_ν in Ref. [8] the following expression was obtained:

$$Z_\nu = 1 + \frac{a_{11}^{(\nu)} g}{\varepsilon} + O(g^2), \quad a_{11}^{(\nu)} = -\frac{(d-1)\overline{S_d}}{8(d+2)}, \quad \overline{S_d} \equiv \frac{S_d}{(2\pi)^d}, \quad (15)$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the d-dimensional sphere of unit radius.

The correlation functions of the renormalized theory do not contain poles in ε. This feature, however, does not solve the problem of finding the infrared asymptotics $s \equiv k/\mu \to 0$, because the corresponding perturbation theory is a series in the parameter $s^{-2\varepsilon}$ growing without limit in the region we are interested in. The problem is solved by passing to the RG representation. To use it for the response functions (9) and (10), rewrite them in the renormalized variables in the form

$$\Gamma_{\varphi\varphi'}(k,\omega=0) = \nu k^2 R_\varphi(s,g),$$

$$\Gamma_{\psi\psi'}(k,\omega=0) = u\nu k^2 R_\psi(s,g,u), \quad (16)$$

where the dimensionless functions R_φ and R_ψ of dimensionless arguments s, g, and u are given by the expressions

$$R_\varphi(s,g) = Z_\nu - \frac{\Sigma_{\varphi'\varphi}(k,\omega=0)}{\nu k^2},$$

$$R_\psi(s,g,u) = Z_\kappa - \frac{\Sigma_{\psi'\psi}(k,\omega=0)}{u\nu k^2}. \quad (17)$$

The RG representation for functions (16) is determined by the relations

$$\Gamma_{\varphi\varphi'}(k,\omega=0) = \overline{\nu} k^2 R_\varphi(s=1,\overline{g}),$$

$$\Gamma_{\psi\psi'}(k,\omega=0) = \overline{u}\overline{\nu} k^2 R_\psi(s=1,\overline{g},\overline{u}), \quad (18)$$

where $\overline{g} = \overline{g}(s,g)$, $\overline{\nu} = \overline{\nu}(s,g,\nu)$, and $\overline{u} = \overline{u}(s,g,u)$ are invariant variables satisfying RG equations of the form

$$[-s\partial_s + \beta_g\partial_g + \beta_u\partial_u - \gamma_\nu \nu\partial_\nu] b(s,g,u) = 0,$$

and normalized by the conditions $\overline{g}(1,g) = g$, $\overline{\nu}(1,g,\nu) = \nu$, and $\overline{u}(1,g,u) = u$. The RG functions β and γ are defined by the renormalization constants according to the relations

$$\beta_g(g) \equiv \mu\partial_\mu|_0 g = g(-2\varepsilon + 3\gamma_\nu), \quad \beta_u(g,u) \equiv \mu\partial_\mu|_0 u = u(\gamma_\kappa - \gamma_\nu),$$

$$\gamma_\nu(g) \equiv \mu\partial_\mu|_0 \ln Z_\nu, \quad \gamma_\kappa(g,u) \equiv \mu\partial_\mu|_0 \ln Z_\kappa, \quad (19)$$

where $\mu\partial_\mu|_0$ denotes the operator $\mu\partial_\mu$ acting at fixed bare parameters g_0, ν_0, and u_0. The last equalities for the β functions in Eq. (19) are a consequence of the connections between the renormalization constants in Eq. (13).

As shown in the one-loop approximation in Refs. [4,8,9], the invariant charges $\overline{g}(s,g)$ and $\overline{u}(s,g,u)$ in the limit $s \to 0$

tend to the infrared-stable fixed point: $\overline{g}(s,g)\to g_*=O(\varepsilon)$, $\overline{u}(s,g,u)\to u_*=O(\varepsilon^0)$, and the invariant viscosity has the powerlike asymptotic behavior

$$\overline{\nu}=\left(\frac{D_0k^{-2\varepsilon}}{\overline{g}}\right)^{1/3}\to\left(\frac{D_0k^{-2\varepsilon}}{g_*}\right)^{1/3}.$$

Thus, the expression for the effective inverse Prandtl number [Eq. (11)] in the inertial range predicted by the RG representation with the account of relations (16) and (18) is

$$u_{eff}=u_*\frac{R_\psi(s=1,g_*,u_*)}{R_\varphi(s=1,g_*)}. \tag{20}$$

The quantity u_{eff} defined by Eq. (11) is universal: the result of its calculation in the inertial range with the aid of relation (20) does not depend on the renormalization scheme. However, different factors on the right-hand side of Eq. (20) do not share this property separately. In particular, in the MS scheme used by us, the quantities R_ψ and R_φ are different from 1; therefore the invariant charge u_* does not coincide with the effective inverse Prandtl number u_{eff}. At the lowest order of perturbation theory, however, the corresponding value $u_*^{(0)}$ is independent of the renormalization scheme, because in this approximation $R_\psi^{(0)}=R_\varphi^{(0)}=1$ and $u_{eff}^{(0)}=u_*^{(0)}$. This feature explains coincidence of values $u_*^{(0)}$ calculated within the RG approach in different renormalization schemes [4,9,10].

IV. TWO-LOOP CALCULATION OF THE PRANDTL NUMBER

The expansion of the functions R_φ and R_ψ [Eq. (17)] in the coupling constant g is of the form

$$R_\varphi=1+g\left[\frac{a_{11}^{(\nu)}}{\varepsilon}-A_\varphi s^{-2\varepsilon}\right]+O(g^2),$$

$$R_\psi=1+g\left[\frac{a_{11}^{(\kappa)}(u)}{\varepsilon}-A_\psi(u)s^{-2\varepsilon}\right]+O(g^2). \tag{21}$$

Here, the quantities A_φ and A_ψ are determined by the one-loop contribution to $\Sigma_{\varphi'\varphi}$ and $\Sigma_{\psi'\psi}$, whereas the coefficients $a_{11}^{(\nu)}$ and $a_{11}^{(\kappa)}$ of representation (14) of the renormalization constants Z_ν and Z_κ are found from the condition of UV finiteness of expressions (21). Substituting relations (21) in Eq. (20), we obtain

$$u_{eff}=u_*\{1+[a_\varphi-a_\psi(u_*)]g_*+O(g_*^2)\}, \tag{22}$$

$$a_\varphi\equiv A_\varphi-\frac{a_{11}^{(\nu)}}{\varepsilon},\quad a_\psi\equiv A_\psi(u_*)-\frac{a_{11}^{(\kappa)}(u_*)}{\varepsilon}. \tag{23}$$

Bearing in mind that $g_*=O(\varepsilon)$, we see that to find u_{eff} at the leading order of the ε expansion, it is enough to know the charge u_* in the one-loop approximation. At the second order, apart from the more accurate values of u_* and g_*, it is necessary to calculate the coefficients a_φ and $a_\psi(u_*)$ of the expansion of the scaling functions (17) and (21) at the leading order in ε as well.

The location of the fixed point (g_*, u_*) is determined by the conditions $\beta_g(g_*)=\beta_u(g_*,u_*)=0$. The nontrivial fixed point with $g_*\neq 0$ is infrared stable [4], and from Eq. (19), the relations

$$\gamma_\nu(g_*)=\frac{2\varepsilon}{3}, \tag{24}$$

$$\gamma_\kappa(g_*,u_*)=\frac{2\varepsilon}{3} \tag{25}$$

follow at this fixed point.

The UV finiteness of the RG functions $\gamma(g,u)$ from Eq. (19) allows us to express them in terms of the coefficient of the first-order pole in ε in expression (14) for the renormalization constants:

$$\gamma=(\beta_g\partial_g+\beta_u\partial_u)\ln Z=-2g\partial_g a_1. \tag{26}$$

The renormalization constant Z_ν at the second order of perturbation theory and the corresponding expression for γ_ν were obtained in Ref. [1]. In particular, the two-loop contribution $a_{21}^{(\nu)}g^2/\varepsilon$ in Z_ν determining the function γ_ν is

$$a_{21}^{(\nu)}=\frac{3(d-1)^2\overline{S_d^2}\lambda}{128(d+2)^2},\quad \overline{S_d}\equiv\frac{S_d}{(2\pi)^d}, \tag{27}$$

from which for g_* [Eqs. (26) and (24)], the result is

$$g_*\overline{S_d}=\frac{8(d+2)\varepsilon}{3(d-1)}(1+\lambda\varepsilon)+O(\varepsilon^3), \tag{28}$$

where

$$\lambda\simeq-1.101,\quad d=3;$$

$$\lambda=-\frac{2}{3(d-2)}+c+O(d-2),\quad d\to 2. \tag{29}$$

From previous analyses, the renormalization constant Z_κ is known in the one-loop approximation only [4]:

$$Z_\kappa=1+\frac{a_{11}^{(\kappa)}g}{\varepsilon}+\left(\frac{C}{\varepsilon^2}+\frac{B}{\varepsilon}\right)(g\overline{S_d})^2+O(g^3),$$

$$a_{11}^{(\kappa)}=-\frac{(d-1)\overline{S_d}}{4du(1+u)}. \tag{30}$$

Here, the notation $C(u)\equiv a_{22}^{(\kappa)}(u)\overline{S_d}^{-2}$ and $B(u)\equiv a_{21}^{(\kappa)}(u)\overline{S_d}^{-2}$ for the coefficients of g^2 in expansion (14) have been introduced for brevity. Their calculation is presented below [it should be noted that, like the one-loop factor $a_{11}^{(\kappa)}$, the two-loop coefficients of the poles in ε in representation (30) are nonpolynomial functions of u]. According to Eq. (26), the RG function γ_κ corresponding to Eq. (30) is

$$\gamma_\kappa=\frac{(d-1)g\overline{S_d}}{2du(1+u)}-4B(g\overline{S_d})^2+O(g^3).$$

An iterative solution of Eq. (25) with respect to u taking into account relation (28) yields

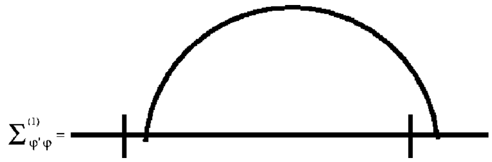

FIG. 1. The one-loop self-energy graph for $\Sigma_{\varphi'\varphi}$. The lines correspond to propagators (6) and (7). Slashes denote the end carrying arguments of the field φ'; plain end carries the arguments of the φ field. Vertices correspond to the factor $V_{ijs}=\mathrm{i}(k_j\delta_{is}+k_s\delta_{ij})$.

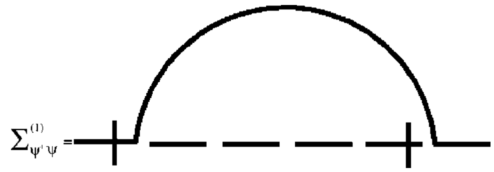

FIG. 2. The one-loop self-energy graph for $\Sigma_{\psi'\psi}$. The lines correspond to propagators (6) and (8). Slashes denote the end carrying arguments of the field ψ'; plain end carries the arguments of the field φ or ψ. Vertices correspond to the factor $V_{ijs}=\mathrm{i}k_j$.

$$u_*=u_*^{(0)}+u_*^{(1)}\varepsilon+O(\varepsilon^2),\quad u_*^{(0)}[1+u_*^{(0)}]=\frac{2(d+2)}{d},\tag{31}$$

$$u_*^{(1)}=\frac{2(d+2)}{d[1+2u_*^{(0)}]}\left[\lambda-\frac{128(d+2)^2}{3(d-1)^2}B(u_*^{(0)})\right].\tag{32}$$

Substituting relations (31) and (32) in Eq. (22) and taking into account Eq. (28) we obtain

$$u_{eff}=u_*^{(0)}\left(1+\varepsilon\left\{\frac{1+u_*^{(0)}}{1+2u_*^{(0)}}\left[\lambda-\frac{128(d+2)^2}{3(d-1)^2}B(u_*^{(0)})\right]+\frac{8(d+2)}{3(d-1)\overline{S}_d}(a_\varphi-a_\psi)\right\}\right)+O(\varepsilon^2).\tag{33}$$

We now turn to the calculation of the constants B, a_ψ and a_φ which determine the Prandtl number. In the one-loop approximation the quantities $\Sigma_{\varphi'\varphi}$ and $\Sigma_{\psi'\psi}$ are represented by the graphs depicted in Figs. 1 and 2, respectively. In these graphs, the lines correspond to propagators (6), (7), and (8) with the convention that ends with slashes corresponds to arguments of the fields φ' and ψ', plain ends of φ and ψ. Vertices in Figs. 1 and 2 correspond to the factors $V_{ijs}=\mathrm{i}(k_j\delta_{is}+k_s\delta_{ij})$ and $\mathrm{i}k_j$, respectively. Upon contraction of indices, integration over time, and introduction of dimensionless wave vector (in units of the external wave vector $\mathbf{p}$) in the integrals we obtain

$$A_\varphi=\frac{1}{2(d-1)}\int\frac{d\mathbf{k}}{(2\pi)^d}\frac{k^{2-d-2\varepsilon}(1-\xi^2)[2k^3\xi-(d-3)k^2-2k(d-1)\xi-(d-1)]}{(2k^2+2k\xi+1)(k^2+2k\xi+1)},\tag{34}$$

$$A_\psi(u)=-\frac{1}{2u}\int\frac{d\mathbf{k}}{(2\pi)^d}\frac{k^{2-d-2\varepsilon}(1-\xi^2)}{(1+u)k^2+2uk\xi+u},\quad \xi\equiv\frac{\mathbf{kp}}{(kp)}.\tag{35}$$

The integrals (34) and (35) are UV divergent in the limit $\varepsilon\to 0$, the residue at the pole is readily found by selecting the asymptotic at $k\to\infty$ contributions to the integrands and discarding the inessential region of integration $k\leqslant 1$. Thus, for the coefficients a_φ and a_ψ together with the renormalization constants Z_ν and Z_κ chosen to cancel divergences in expressions (23), we find

$$\frac{a_{11}^{(\nu)}}{\varepsilon}=\frac{1}{4(d-1)(2\pi)^d}\int_1^\infty\frac{dk}{k^{1+2\varepsilon}}\int d\hat{\mathbf{k}}(1-\xi^2)\times(2k\xi-d+3-6\xi^2),$$

$$\frac{a_{11}^{(\kappa)}(u)}{\varepsilon}=\frac{-1}{2u(1+u)(2\pi)^d}\int_1^\infty\frac{dk}{k^{1+2\varepsilon}}\int d\hat{\mathbf{k}}(1-\xi^2),\quad \hat{\mathbf{k}}\equiv\frac{\mathbf{k}}{k}.$$

Replacing the integral over directions of the unit vector $\hat{\mathbf{k}}$ by the average over its directions $\int d\hat{\mathbf{k}}\cdots=S_d\langle\cdots\rangle$ and taking into account that

$$\langle\xi^{2n}\rangle=\frac{(2n-1)!!}{d(d+2)\cdots(d+2n-2)},\quad \langle\xi^{2n+1}\rangle=0,\tag{36}$$

we arrive at result (15) for $a_{11}^{(\nu)}$ and (30) for $a_{11}^{(\kappa)}$. In view of the preceding argumentation, the coefficients a_φ and a_ψ in Eq. (23) at the leading order in ε may be written as

$$a_\varphi=\frac{1}{4(d-1)(2\pi)^d}\int_0^\infty dk\int d\hat{\mathbf{k}}(1-\xi^2)\times\left\{\frac{2k[2k^3\xi-(d-3)k^2-2k(d-1)\xi-(d-1)]}{(2k^2+2k\xi+1)(k^2+2k\xi+1)}-\frac{\theta(k-1)(2k\xi-d+3-6\xi^2)}{k}\right\},\tag{37}$$

$$a_\psi=\frac{-1}{2u(2\pi)^d}\int_0^\infty dk\int d\hat{\mathbf{k}}(1-\xi^2)\times\left[\frac{k}{(1+u)k^2+2uk\xi+u}-\frac{\theta(k-1)}{k(1+u)}\right].\tag{38}$$

At $d=3$ from relations (37) and (38), we obtain

$$a_{\varphi}=\frac{\overline{S_3}}{8}\int_0^{\infty}dk\int_{-1}^{1}d\xi(1-\xi^2)\left\{\frac{2k[k^3\xi-2k\xi-1]}{(2k^2+2k\xi+1)(k^2+2k\xi+1)}-\theta(k-1)\left(\xi-\frac{3\xi^2}{k}\right)\right\}, \tag{39}$$

$$a_{\psi}=\frac{-\overline{S_3}}{4u}\int_0^{\infty}dk\int_{-1}^{1}d\xi(1-\xi^2)\left[\frac{k}{(1+u)k^2+2uk\xi+u}-\frac{\theta(k-1)}{k(1+u)}\right],\quad d=3,\quad \overline{S_3}=\frac{1}{2\pi^2}, \tag{40}$$

Numerical evaluation of integrals (39) and (40) with $u=u_*^{(0)}$ from Eq. (31) yields

$$a_{\varphi}=-0.047718\overline{S_3},\quad a_{\psi}=-0.04139\overline{S_3}. \tag{41}$$

It is convenient to find the two-loop contributions to the renormalization constant Z_κ from the condition that the quantity R_ψ from Eq. (17) is UV finite in the limit $k\to 0$. In terms of the reduced quantity

$$\Sigma\equiv\lim_{k\to 0}\frac{\Sigma_{\psi',\psi}(\omega=0,k)}{u\nu k^2}, \tag{42}$$

this condition may be cast in the form

$$Z_\kappa(\varepsilon)-\Sigma(\varepsilon)=O(\varepsilon^0). \tag{43}$$

The limit $k\to 0$ in expression (42) does exist, provided the IR regularization of the graphs has been taken care of. In the MS scheme renormalization constants do not depend on the method of such regularization. With our choice of the pumping function (12) it is accomplished by the cutoff of the propagator $\langle\varphi\varphi\rangle_0$ (6) at $k<m$.

Let us choose further the wave vector of integration such that in the lines $\langle\varphi\varphi\rangle_0$ it flows alone (for the graphs $\Sigma_{\psi',\psi}$ such a choice is always possible). Integration over all the wave numbers will then be carried out within the limits from m to ∞.

The one-loop contribution to Σ is determined by the graph of Fig. 2 as

$$\Sigma^{(1)}=-\frac{g\mu^{2\varepsilon}}{2u(Z_\nu+uZ_\kappa)Z_\nu}\int\frac{d\mathbf{k}}{(2\pi)^d}\frac{(1-\xi^2)\theta(k-m)}{k^{d+2\varepsilon}}$$
$$=-\frac{g\mu^{2\varepsilon}}{2u(Z_\nu+uZ_\kappa)Z_\nu(2\pi)^d}\int_m^{\infty}\frac{dk}{k^{d+2\varepsilon}}\int d\hat{\mathbf{k}}(1-\xi^2)$$
$$=-\frac{g\overline{S_d}\mu^{2\varepsilon}}{2u(Z_\nu+uZ_\kappa)Z_\nu}\int_m^{\infty}\frac{dk}{k^{d+2\varepsilon}}\langle(1-\xi^2)\rangle,$$

which, together with relations (15), (30), and (36) yields

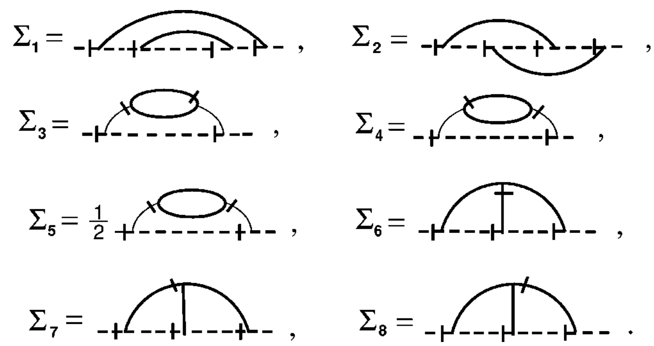

FIG. 3. The two-loop self-energy graphs for $\Sigma_{\psi'\psi}$. The lines correspond to propagators (6), (7), and (8). Slashes denote the end carrying arguments of the field φ' or ψ'; plain end carries the arguments of the field φ or ψ. Vertices correspond to the factor $V_{ijs}=\mathrm{i}(k_j\delta_{is}+k_s\delta_{ij})$ or $V_{ijs}=\mathrm{i}k_j$.

$$\Sigma^{(1)}=-\frac{g\overline{S_d}(d-1)\tau^{-2\varepsilon}}{4\varepsilon u(Z_\nu+uZ_\kappa)Z_\nu d}$$
$$=-\frac{g\overline{S_d}(d-1)\tau^{-2\varepsilon}}{4\varepsilon u(1+u)d}\times\left\{1-[ua_{11}^{(\kappa)}+(2+u)a_{11}^{(\nu)}]\frac{g\overline{S_d}}{\varepsilon(1+u)}\right\}+O(g^3), \tag{44}$$

where $\tau\equiv m/\mu$. Extracting the pole contributions in ε from expressions (44) we obtain

$$\Sigma^{(1)}=-\frac{g\overline{S_d}(d-1)}{4\varepsilon u(1+u)d}\left\{1-[ua_{11}^{(\kappa)}+(2+u)a_{11}^{(\nu)}]\times\frac{g\overline{S_d}}{(1+u)}\left(\frac{1}{\varepsilon}-2\ln\tau\right)\right\}+O(\varepsilon^0). \tag{45}$$

Substituting relation (45) in Eq. (43) and requiring cancelation of pole contributions in the linear in g approximation, we return to expression (30) for $a_{11}^{(\kappa)}$. The terms of order g^2 are required for the calculation of the renormalization constant in the two-loop approximation.

The two-loop contribution $\Sigma^{(2)}$ to the self-energy operator $\Sigma_{\psi'\psi}$ is determined by the sum of the graphs depicted in Fig. 3 [normalization according to Eq. (42) is implied]. When substituting propagators (6)–(8)—expressed in terms of the renormalized variables—in the graphs of Fig. 3 it is possible to put $Z_\nu=Z_\kappa=1$ with the necessary accuracy. Contracting indices and integrating over time, we obtain

$$\Sigma_n=\frac{(g\overline{S_d})^2\mu^{4\varepsilon}}{192uv^2}\int_m^{\infty}\frac{dk}{k^{1+2\varepsilon}}\int_m^{\infty}\frac{dq}{q^{1+2\varepsilon}}\int_{-1}^{1}d\xi\times\frac{(1-\xi^2)J_n}{[v(k^2+q^2)+ukq\xi]},\quad v\equiv\frac{1+u}{2},\quad n=1,2, \tag{46}$$

where

$$J_1 = 2q^2, \quad J_2 = -zkq, \tag{47}$$

and

$$\Sigma_n = \frac{(g\overline{S_d})^2\mu^{4\varepsilon}}{96uv}\int_m^\infty \frac{dk}{k^{1+2\varepsilon}}\int_m^\infty \frac{dq}{q^{1+2\varepsilon}}\int_{-1}^{1} d\xi \frac{(1-\xi^2)q^2 J_n}{k^2+2kq\xi+q^2},$$
$$n = 3\cdots 8, \tag{48}$$

with

$$J_3 = k(k^3+2k^2q\xi - q^3\xi)\left[\frac{1}{(k^2+kq\xi+q^2)(vk^2+kq\xi+q^2)} + \frac{1}{vk^2(vk^2+kq\xi+q^2)} + \frac{1}{k^2(k^2+kq\xi+q^2)}\right], \tag{49}$$

$$J_4 = \frac{(k^3+2k^2q\xi-q^3\xi)}{k(k^2+kq\xi+q^2)}, \tag{50}$$

$$J_5 = -\frac{k^2[k^4+q^4+kq\xi(k^2+q^2)]}{(k^2+q^2+kq\xi)(k^2+q^2+2kq\xi)}\left[\frac{2}{k^2+2kq\xi+q^2} + \frac{1}{v(k^2+q^2)+ukq\xi}\right], \tag{51}$$

$$J_6 = \frac{kq\xi(k^2-q^2)}{2(k^2+kq\xi+vq^2)}\left(\frac{1}{vq^2}+\frac{1}{k^2+kq\xi+q^2}\right), \tag{52}$$

$$J_7 = -\frac{k^3(2k^3+3k^2q\xi-q^3\xi)}{2(k^2+2kq\xi+q^2)}\left\{\frac{1}{vk^2[v(k^2+q^2)+ukq\xi]} + \frac{1}{vk^2(vk^2+kq\xi+q^2)} + \frac{1}{(k^2+kq\xi+q^2)(vk^2+kq\xi+q^2)}\right\}, \tag{53}$$

$$J_8 = \frac{k(2k^3+3k^2q\xi-q^3\xi)}{2(k^2+kq\xi+q^2)[v(k^2+q^2)+ukq\xi]}. \tag{54}$$

Integrals (46)–(54) may be represented as

$$\Sigma_i = \mu^{4\varepsilon}(g\overline{S_d})^2\int_m^\infty \frac{dk}{k^{1+2\varepsilon}}\int_m^\infty \frac{dq}{q^{1+2\varepsilon}}\int_{-1}^{1} d\xi f_i(\xi,k/q), \tag{55}$$

or, after the corresponding stretching of integration variables, as

$$\Sigma_i = (g\overline{S_d})^2\int_\tau^\infty \frac{dk}{k^{1+2\varepsilon}}\int_\tau^\infty \frac{dq}{q^{1+2\varepsilon}}\int_{-1}^{1} d\xi f_i(\xi,k/q), \quad \tau \equiv m/\mu, \tag{56}$$

or, finally, as

$$\Sigma_i = \sigma_i(g\overline{S_d})^2\tau^{-4\varepsilon}, \quad \sigma_i \equiv \int_1^\infty \frac{dk}{k^{1+2\varepsilon}}\int_1^\infty \frac{dq}{q^{1+2\varepsilon}}\int_{-1}^{1} d\xi f_i(\xi,k/q). \tag{57}$$

We are interested in the coefficients of the pole contributions to $\Sigma_i(\varepsilon)$:

$$\sigma_i = \frac{c_i}{\varepsilon^2} + \frac{b_i}{\varepsilon} + O(\varepsilon^0),$$

$$\Sigma_i = (g\overline{S_d})^2\left[\frac{c_i}{\varepsilon^2} + \frac{b_i - 4c_i\ln\tau}{\varepsilon} + O(\varepsilon^0)\right]. \tag{58}$$

For the functions $f_i(z,k/q)$ with $i=2,5\cdots 8$ the equations $f_i(z,0)=f_i(z,\infty)=0$ hold, revealing that integrals over k and q in Eq. (55) are separately convergent, so that the divergence at $\varepsilon\to 0$ in the corresponding Σ_i is brought about by the region, in which k and q tend to infinity simultaneously. As a consequence, the second-order pole is absent in such Σ_i: $c_i=0$ for $i=2,5\cdots 8$.

For Σ_i with $i=1,3,4$ $f_i(z,\infty)=0$ as before, which means absence of divergence in the integral over k in Eq. (55). For these graphs, however, $f_i(z,0)=\text{const}\neq 0$, so that the integral over q diverges at $\varepsilon\to 0$ leading to the appearance of the pole of second order in the full integral.

Expressions (56) may be simplified with the use of the identity

$$\Sigma_i = -\frac{\tau\partial_\tau\Sigma_i}{4\varepsilon} \tag{59}$$

following from Eq. (57). Calculating the right-hand side of Eq. (59) with the aid of relations (56) and introducing the dimensionless integration variables, we obtain

$$\Sigma_i = \frac{\tau^{-4\varepsilon}(g\overline{S_d})^2}{4\varepsilon}\int_1^\infty \frac{d\kappa}{\kappa^{1+2\varepsilon}}\int_{-1}^{1} d\xi[f_i(\xi,\kappa)+f_i(\xi,1/\kappa)]. \tag{60}$$

This operation has reduced the number of iterated integrations and allowed for explicit extraction one pole in ε. For $i=2,5\cdots 8$, the integral in Eq. (60) is finite for $\varepsilon=0$ and determines the residue of the first-order pole:

$$c_i = 0, \quad b_i = \frac{1}{4}\int_1^\infty \frac{d\kappa}{\kappa}\int_{-1}^{1} d\xi[f_i(\xi,\kappa)+f_i(\xi,1/\kappa)],$$
$$i = 2,5\cdots 8. \tag{61}$$

For Σ_i with $i=1,3,4$, the coefficient of the second-order pole is obtained by the replacement of the function $[f_i(\xi,\kappa)+f_i(\xi,1/\kappa)]$ in the integrand in Eq. (60) by its limiting value at $\kappa\to\infty$: $f_i(\xi,\infty)+f_i(\xi,0)=f_i(\xi,0)$ [we recall that $f_i(z,\infty)=0$]. Integration over κ then becomes trivial, which yields

$$c_i = \frac{1}{8}\int_{-1}^{1} d\xi f_i(\xi,0), \quad i = 1,3,4. \tag{62}$$

The remaining integral with the change $f_i(\xi,\kappa)\to[f_i(\xi,\kappa)-f_i(\xi,0)]$ is finite at $\varepsilon=0$ and determines the residue of the first-order pole:

TABLE I. Residues of the first-order poles in ε of the dimensionless integrals (57) corresponding to the two-loop graphs of Fig. 3.

i	1	2	3	4	5	6	7	8
$b_i \times 10^3$	0.1099	0.0944	0.8691	0.0057	−3.9382	0.0672	−1.9647	0.5899

$$b_i = \frac{1}{4}\int_1^\infty \frac{d\kappa}{\kappa}\int_{-1}^1 dz[f_i(z,\kappa) + f_i(z,1/\kappa) - f_i(z,0)], \quad i = 1,3,4. \tag{63}$$

Let us write condition (43) at order g^2 for $d=3$. With the use of the corresponding terms of the one-loop contribution (45), the summed two-loop contributions (58) and expression (30) for the renormalization constant Z_κ, we obtain

$$\frac{C}{\varepsilon^2} + \frac{B}{\varepsilon} = \frac{1}{6\varepsilon u(1+u)^2\overline{S_d}}[ua_{11}^{(\kappa)} + (2+u)a_{11}^{(\nu)}]\left(\frac{1}{\varepsilon} - 2\ln\tau\right) + \sum_1^8\left(\frac{c_i}{\varepsilon^2} + \frac{b_i - 4a_i \ln\tau}{\varepsilon}\right). \tag{64}$$

With the aid of expressions (46)–(50) and (55) in Eq. (62), it is not difficult to find

$$c_1 = \frac{1}{72u(1+u)^3}, \quad c_3 = \frac{(3+u)}{480u(1+u)^2},$$

$$c_4 = \frac{1}{480u(1+u)}, \quad d = 3.$$

Substituting these values in Eq. (64) and taking into account relations (30) and (15) for $a_{11}^{(\kappa)}$ and $a_{11}^{(\nu)}$, we see that the terms with $\ln\tau$ in Eq. (64) are automatically canceled (as a consequence of renormalizability of the model), whereas for the coefficient C of the second-order pole we obtain

$$C = -\frac{3u^2 + 9u + 16}{720u(1+u)^3}.$$

For the coefficients b_i numerical integration of expressions (61) and (63) with $u = u_*^{(0)}$ from Eq. (31) yields the results quoted in Table I, which for the coefficient B in Eq. (64) lead to the value

$$B(u_*^{(0)}) = \sum_{i=1}^8 b_i = -4.1666 \times 10^{-3}.$$

Substituting this value in Eq. (33) as well as a_φ and a_ψ from Eq. (41) and λ from Eq. (29), we obtain the final expression for the effective inverse Prandtl number:

$$u_{eff} = u_*^{(0)}(1 - 0.0358\varepsilon) + O(\varepsilon^2),$$

$$u_*^{(0)} = \frac{\sqrt{43/3} - 1}{2} \simeq 1.3930, \quad d = 3. \tag{65}$$

At the physical value $\varepsilon = 2$, this yields for the turbulent Prandtl number Pr_t the result

$$\mathrm{Pr}_t^{(0)} \simeq 0.7179, \quad \mathrm{Pr}_t \simeq 0.7693, \tag{66}$$

in one-loop and two-loop accuracy, respectively.

As seen from Eqs. (65) and (66), the two-loop correction to the Prandtl number is small, even for the real value $\varepsilon=2$. This smallness is a consequence of a nearly complete cancelation of two large contributions: the term proportional to λ in the brackets of Eq. (33), determined by the renormalization constant of viscosity Z_ν [see Eq. (27)], and the term proportional to B, determined by the renormalization constant Z_κ of the diffusion coefficient (30). Indeed, at $d=3$, the second term is equal to $-800B/3 \simeq 1.111$, whereas, according to (29), $\lambda \simeq -1.101$, and thus the whole expression in brackets is equal to $\lambda - 800B/3 \simeq 0.010$; i.e., by two orders smaller than each term separately. Inspection of individual graphs of Fig. 3 determining the quantity B reveals that the largest contribution is due to the graph Σ_5, and the corresponding coefficient b_5 (see Table I) is close to the value of the whole sum $B = \sum_{i=1}^8 b_i$. Analysis shows that Σ_5 is the only graph possessing a singularity at $d \to 2$: $b_5 \simeq -1/1024(d-2)$. A similar situation was met in the calculation of the quantity λ in Ref. [1]: the largest contribution was given by a graph having a pole in $d-2$. As follows from (28), the coefficient of the singular contribution to λ is such that as a whole the two-loop contribution to u_{eff} [see Eq. (33)] turns out to be finite at $d=2$. Let us also point out significant compensation in relation (33) of smaller in magnitude and finite at $d=2$ contributions in the expression $a_\varphi - a_\psi$ [see Eq. (41)].

V. CONCLUSION

In the present problem—as is usual in perturbative field theories [11]—we are dealing with asymptotic (semiconvergent) series with the typical factorial growth of the number of graphs with the order of perturbation theory (the anticipated growth of the number of graphs has been explicitly demonstrated for the simpler model of passive advection in a given quenched Gaussian random velocity field in Ref. [12]). Semiconvergent series may be and are used for numerical estimates as long as the magnitude of the correction of each subsequent order is much less than that of the preceding order—a property which for such a series is bound to break down at some order, which, however, is not known *a priori*. The most notable physical demonstration of the usefulness of semiconvergent series is the quantum electrodynamics. In the ε expansion of critical exponents, different situations have been met with both small and large corrections to leading-order values [13].

The main conclusion to be drawn from the two-loop value of the effective inverse Prandtl number (65) obtained in the present paper is that the correction term is strikingly small. Even at the real value $\varepsilon=2$, it is only 7% of the leading contribution. Apparently, this is the reason for the favorable comparison of the one-loop value of the turbulent Prandtl number 0.72 [3,4] with the experiment. This result is already in the range 0.7–0.9 of measured values established quite a while ago [5]. Fairly recent experimental results in circular turbulent jets emphasize the midpoint of this range: for the region of approximately constant turbulent Prandtl number, the value 0.81 ± 0.05 is found in Ref. [6], whereas in Ref. [7] a recommended value 0.8 for turbulence modeling in high Reynolds-number flows is put forward on the basis of the results in the region of slight variation of the turbulent Prandtl number in the range 0.7–0.9. In view of these numbers, we are inclined to conclude that the already fairly good one-loop result is improved by the two-loop correction, whose account (66) leads to the value 0.77 for the turbulent Prandtl number.

The obtained result is somewhat unexpected: similar two-loop corrections to the Kolmogorov constant and the skewness factor are large [1]. When corrections are not small, knowledge of large-order asymptotic behavior of the series is required to construct resummation schemes useful for numerical estimates. In the theory of *static* critical phenomena, the instanton approach together with Borel summation has been widely used to this end [11]. In case of dynamic models, however, only first steps have been made in this direction [14].

The summation used in the calculation of the Kolmogorv constant in Ref. [2] differs from the traditional Borel summation in that it is based on an approximate calculation of *all* high-order terms in ε expansion (apart from exactly calculated first two) with the account of the leading terms of their Laurent expansion in $d-2$. The success of such a summation is related to the specific property of the ε expansion in the theory of turbulence: the presence of poles in $d-2$ in a certain class of graphs and the significant contribution of these graphs at $d=3$. Our two-loop calculation of the Prandtl number has shown that such graphs exist in this problem as well, but their poles in $d-2$ cancel each other, which might an explanation of the first correction term in the ε expansion of the Prandtl number.

Thus, our results complement the conclusion made in Refs. [1,2]. In the two-loop approximation the main contribution is due to graphs having a singularity at $d=2$ and it is necessary to sum such graphs. For quantities in which this singularity is absent the two-loop contribution is relatively small and the results of the usual ε expansion appear fairly reliable at the level of accuracy suggested by the two-loop correction.

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